On generalizations of Verlinde's formula

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Abstract

It is shown how traces of higher genus mapping classes of finite order may be expressed by Verlinde-like formulae involving only genus one modular matrix elements. The 3D topological argument leading to the result is explained, and the related trace identities for modular matrix elements are presented.
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One of the basic results of 2D conformal field theory is the celebrated formula of Verlinde [1]:

$$N_{pqr} = \sum_s \frac{S_{ps} S_{qs} S_{rs}}{S_{0s}} \tag{1}$$

(as usual, we label by 0 the vacuum of the theory), expressing the fusion rule coefficient N_{pqr} in terms of the matrix elements of the modular transformation $S : \tau \to -1/\tau$. Denoting by $\mathcal{V}_g(p_1, \ldots, p_n)$ the space of genus g holomorphic n-point blocks with the insertions $p_1, \ldots, p_n$, and remembering that $N_{pqr} := \dim \mathcal{V}_0(p, q, r)$, Eq. (1) implies the more general result [2]:

$$\dim \mathcal{V}_g(p_1, \ldots, p_n) = \sum_q S_{0q}^{2-2g} \prod_{i=1}^n \frac{S_{qp_i}}{S_{0q}}. \tag{2}$$

Eqs. (1) and (2) are not only of importance for physics, but they have raised much attention in the mathematical literature as well [3,4].

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The virtue of Verlinde's formula is that it expresses the dimension of the space $\mathcal{V}_g(p_1, \ldots, p_n)$ of holomorphic blocks, or what is the same, the trace of the identity operator acting on $\mathcal{V}_g(p_1, \ldots, p_n)$, in terms of seemingly unrelated quantities, matrix elements of modular transformations acting on the space of genus 1 characters. But besides the identity, there are other operators of interest that act naturally on $\mathcal{V}_g(p_1, \ldots, p_n)$: the operators representing the transformations of the mapping class group $M_{g,n}$ of genus g surfaces with n punctures. It seems therefore reasonable to ask whether there exists Verlinde-like expressions for the traces of mapping classes other than the identity. This question has been addressed previously in [5], and it was found that there exists such expressions for some mapping classes. The following problems have been left open by that work:

(1) Characterize those mapping classes for which there exist a Verlinde-like formula expressing the trace in terms of the matrix elements of modular transformations.
(2) Give a simple recipe to write down the trace formula in case there is one.

The purpose of the present note is to solve the above problems in the case of closed surfaces, i.e. for $n = 0$.

Before giving the answers and sketching the argument leading to them, let us point out an important consequence already noticed in [5]. Namely, one gets trace formulae already at genus 1, i.e. for modular transformations acting on the space of genus 1 characters, resulting in non-trivial algebraic identities for modular matrix elements, e.g. one has

$$\mathrm{Tr}(S) = \sum_p \frac{(S^{-1}T^4S)_{p0}(S^{-1}T^4S)_{p0}(S^{-1}T^{-2}S)_{p0}}{S_{p0}}, \tag{3}$$

which should be compared with the obvious answer:

$$\mathrm{Tr}(S) = \sum_p S_{pp}. $$

The equality of the two expressions for $\mathrm{Tr}(S)$ may be shown to be independent of the usual consistency requirements on modular matrix elements, i.e. Verlinde's theorem and the modular relation, and lends itself to a simple numerical check. The relevant trace identities arising this way will be summarized later in Table 1.

Let us now turn to the actual topic of this note, i.e. giving a simple characterization of those mapping classes whose trace may be expressed by a Verlinde-like formula. As we shall argument later, there is such a trace formula in case the mapping class has a fixed point in its action on Teichmuller-space. By a theorem of Nielsen [6], this is equivalent to the assertion that the mapping class has finite order. In particular for $g = 1$ this means that we get trace formulae for the powers of S and ST, leading to non-trivial trace identities.

To give a simple answer to the second problem, we have to first introduce the relevant mathematical concept, namely that of twisted dimensions [7], which are the basic building blocks of the fusion rules of permutation orbifolds.

First, for a rational number $r = k/n$ in reduced form, i.e. with positive denominator $n > 0$ and with $\gcd(k, n) = 1$, we define the matrix $\Lambda(r)$, with matrix elements:

$$\Lambda_{pq}(r) = T_{pp}^{-r} M_{pq} T_{qq}^{-r^*}, \tag{4}$$

Table 1
Trace formulae for $g = 1$

	Mapping class		
	S	ST	$(ST)^2$
Fixed point	$\tau = i$	$\tau = \exp\left(\dfrac{2\pi i}{3}\right)$	$\tau = \exp\left(\dfrac{2\pi i}{3}\right)$
Lift	$z \mapsto iz$	$z \mapsto \exp\left(\dfrac{\pi i}{3}\right)z$	$z \mapsto \exp\left(\dfrac{2\pi i}{3}\right)z$
N	4	6	3
Signature	$(0; \frac{1}{4}, \frac{1}{4}, \frac{1}{2})$	$(0; \frac{1}{6}, \frac{1}{3}, \frac{1}{2})$	$(0; \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
Trace	$\mathcal{D}_0 \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$	$\mathcal{D}_0 \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}$	$\mathcal{D}_0 \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$

where M_{pq} denotes the matrix elements of the modular transformation:

$$M = \begin{pmatrix} k & y \\ n & x \end{pmatrix}$$

with x, y any integer solution of the equation $kx - ny = 1$—such solutions always exist because $\gcd(k, n) = 1$—and $r^* = x/n$. It may be shown that the matrix $\Lambda(r)$ is well defined (after choosing a definite branch of the logarithm), and that it is periodic in r, i.e. $\Lambda(r + 1) = \Lambda(r)$. Notice that $\Lambda(0) = S$, and more generally

$$\Lambda\left(\frac{1}{n}\right) = T^{-1/n} S^{-1} T^{-n} S T^{-1/n}.$$

We can now define the twisted dimensions as

$$\mathcal{D}_g \begin{pmatrix} p_1 & \cdots & p_n \\ r_1 & \cdots & r_n \end{pmatrix} = \sum_q S_{0q}^{2-2g} \prod_{i=1}^{n} \frac{\Lambda_{qp_i}(r_i)}{S_{0q}} \tag{5}$$

for a non-negative integer g—the genus—a sequence of primaries $p_1, \ldots, p_n$ and a sequence of rationals $r_1, \ldots, r_n$ (the characteristics). In case all the characteristics are zero, we have

$$\mathcal{D}_g \begin{pmatrix} p_1 & \cdots & p_n \\ 0 & \cdots & 0 \end{pmatrix} = \dim \mathcal{V}_g(p_1, \ldots, p_n) \tag{6}$$

by Verlinde's formula

After these preliminaries, let us present the solution of the second problem. According to what has been said above, there is a Verlinde-like formula for the trace in case the mapping class $\gamma \in M_{g,0}$ has a fixed point in its action on the Teichmuller-space $\mathcal{X}_g$. Such a fixed point τ corresponds to a closed genus g surface $\mathcal{S}$ with a non-trivial automorphism group, and the mapping class γ lifts to an automorphism $\hat{\gamma} \in \text{Aut}(\mathcal{S})$ of finite order N—actually, one may identify $\text{Aut}(\mathcal{S})$ with the stabilizer of τ in $M_{g,0}$ [8]. Dividing out $\mathcal{S}$ by the action

of $\hat{\gamma}$ we get a new surface $\mathcal{S}/\hat{\gamma}$ of genus g^*. Of all the orbits of $\hat{\gamma}$ on $\mathcal{S}$, only a finite number have non-trivial stabilizer subgroups, whose orders we denote by $n_1, \ldots, n_r$—these orbits correspond to the ramification points of the holomorphic covering map $\pi : \mathcal{S} \to \mathcal{S}/\hat{\gamma}$. In particular, each n_i divides N. All these quantities are related by the Riemann–Hurwitz formula:

$$2g - 2 = N\left(2g^* - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{n_i}\right)\right). \tag{7}$$

The action of $\hat{\gamma}$ near the ith branch point is given by

$$\hat{\gamma} : z \mapsto \exp\left(2\pi i \frac{k_i}{n_i}\right) z$$

for some integer $0 \le k_i < n_i$ coprime to n_i, where z is a local coordinate. Thus the ratio k_i/n_i gives the monodromy of the covering map $\pi : \mathcal{S} \to \mathcal{S}/\hat{\gamma}$ around the corresponding branch point.

We are now in position to give a closed expression for the traces. The claim is that the trace of γ on $\mathcal{V}_g$ is given by

$$\mathrm{Tr}(\gamma) = \mathcal{D}_{g^*}\begin{pmatrix} 0 & \cdots & 0 \\ \dfrac{k_1}{n_1} & \cdots & \dfrac{k_r}{n_r} \end{pmatrix}. \tag{8}$$

Let us illustrate the above results for ordinary modular transformations at genus 1. In this case Teichmuller-space is nothing but the complex upper half-plane $\mathcal{X}_1 = \{\tau \in \mathbb{C} | \mathrm{Im}(\tau) > 0\}$, and the mapping class group is $M_{1,0} = \mathrm{SL}(2, \mathbb{Z})$, acting on $\mathcal{X}_1$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \mapsto \frac{a\tau + b}{c\tau + d}.$$

Note that

$$S^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

acts trivially on $\mathcal{X}_1$, i.e. each $\tau \in \mathcal{X}_1$ is a fixed point of it. The lift of S^2 is $z \mapsto -z$, which is clearly an involutive automorphism with fixed points $\{1, 1/2, \tau/2, (1+\tau)/2\}$. According to Eq. (8) we should have

$$\mathrm{Tr}(S^2) = \mathcal{D}_0\begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \tag{9}$$

The expression on the rhs of Eq. (9) may be rewritten as $\sum_p \nu_p^2$, where ν_p denotes the Frobenius–Schur indicator of the primary p—which is $+1$ for real, -1 for pseudo-real, and 0 for complex primaries [10]—and the equality of this last expression with the trace of S^2 follows at once from $(S^2)_{pp} = \nu_p^2$. While we get nothing new, we have an instance where the trace formula may be derived by other means.

The other mapping classes of finite order are

(1) $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, whose fixed point is $\tau = \mathrm{i}$.

(2) $ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, whose fixed point is $\tau = \exp(2\pi\mathrm{i}/3)$.

(3) $(ST)^2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$, with fixed point $\tau = \exp(2\pi\mathrm{i}/3)$.

(4) The inverses of the above, which will not give anything new since $\mathrm{Tr}(X^{-1}) = \overline{\mathrm{Tr}(X)}$ by the unitarity of the modular representation.

We have to lift the above mapping classes to automorphisms of their fixed points. Simple considerations show that the lifts are, respectively:

(1) $z \mapsto \mathrm{i}z$, of order $N = 4$.
(2) $z \mapsto \exp(\pi\mathrm{i}/3)z$, of order $N = 6$.
(3) $z \mapsto \exp(2\pi\mathrm{i}/3)z$, of order $N = 3$.

It is straightforward to enumerate the fixed points of the above transformations (acting on the corresponding tori), and to deduce from this the data $(g^*; k_1/n_1, \ldots, k_r/n_r)$. The results are summarized in Table 1, where we have expressed the traces in terms of twisted dimensions. Should we express the relevant twisted dimension in terms of modular matrix elements, we would get Eq. (3) for the trace of S (and similar results in the other cases). Equating the resulting expressions with the obvious one for the corresponding modular transformation, we get the list of all non-trivial trace identities that should hold in any consistent RCFT.

Let us now turn to the origin of the trace formulae. For this we need to recall that to any RCFT there corresponds a 3D topological field theory [9], and that this later assigns a complex number $Z(M)$—the partition function—to each three-dimensional closed manifold, such that $Z(M_1) = Z(M_2)$ whenever M_1 and M_2 are homeomorphic. The partition function of a given 3-manifold can be determined in principle from the knowledge of the modular matrix elements of the RCFT via surgery, although this is by no means a simple task in general. The point is that for some classes of 3-manifolds one can give closed expressions for the partition function.

One such class is that of the so-called fibred manifolds [11]. These are obtained by the following procedure: take a closed surface $\mathcal{S}$ of genus g, and form the product $\mathcal{S} \times [0, 1]$. The resulting 3-manifold is not closed, as it has two boundary components, each homeomorphic to $\mathcal{S}$, so we have to glue together these boundary components. When doing so, one has the freedom to identify the boundary components via a self-homeomorphism $\gamma : \mathcal{S} \to \mathcal{S}$. As it turns out, the topological equivalence class of the resulting closed 3-manifold does not depend on the actual choice of γ, but only on its mapping class $[\gamma]$. This implies that for each $\gamma \in M_{g,0}$ we get a closed 3-manifold F_γ, well-defined up to topological equivalence. It might not come as a big surprise that the partition function of F_γ is just the trace of the

operator representing γ on the space of genus g characters, i.e.:

$$Z(F_\gamma) = \text{Tr}(\gamma). \tag{10}$$

There is another class of 3-manifolds for which we know the partition function, the Seifert-manifolds. They may be constructed according to the following recipe: consider a closed (oriented) surface $\mathcal{S}$ of genus g, and cut out n non-overlapping disks to obtain a surface $\mathcal{S}'$. The product $\mathcal{S}' \times S^1$ is not closed, its boundary consisting of n disjoint 2-tori. To get a closed 3-manifold, one has to paste in solid tori to these boundary components, and in doing so, one has to glue the ith boundary component to the boundary of the corresponding solid torus by means of a modular transformation M_i. As it turns out, the resulting Seifert-manifold $S(g; r_1, \ldots, r_n)$ is characterized besides the genus g, by the sequence $r_i = M_i(i\infty)$ of rationals modulo integers, i.e. the images of the cusp at infinity. The partition function of a Seifert-manifold [12,13] expressed in terms of twisted dimensions reads

$$Z[S(g; r_1, \ldots, r_n)] = \mathcal{D}_g \begin{pmatrix} 0 & \cdots & 0 \\ r_1 & \cdots & r_n \end{pmatrix}. \tag{11}$$

We are nearly done, all that remains is to notice that the class of Seifert-manifolds and that of fibred manifolds overlap. Actually, a fibred manifold F_γ is a Seifert-manifold exactly when the mapping class γ has a fixed point in its action on Teichmuller-space, i.e. when it has finite order [6]. So for such $\gamma \in M_{g,0}$, F_γ is homeomorphic to $S(g^*; r_1, \ldots, r_n)$ for some signature $(g^*; r_1, \ldots, r_n)$, consequently their partition functions coincide, i.e.:

$$\text{Tr}(\gamma) = \mathcal{D}_{g^*} \begin{pmatrix} 0 & \cdots & 0 \\ r_1 & \cdots & rn \end{pmatrix}.$$

It remains to determine the signature $(g^*; r_1, \ldots, r_n)$ corresponding to a given mapping class γ of finite order. The detailed analysis [6] leads to the result presented earlier: for $\gamma \in M_{g,0}$ of finite order, F_γ is homeomorphic to a Seifert-manifold whose signature is determined by the monodromy of the covering map $\pi : \mathcal{S} \to \mathcal{S}/\hat{\gamma}$.

One may look at the above results from a different perspective. Let $g > 1$, and let us take some point $\tau \in \mathcal{X}_g$ in Teichmuller-space, and suppose that the corresponding Riemann-surface $\mathcal{S}$ has a non-trivial automorphism group $\mathcal{A}$ (which is known to be finite). To each element of $\mathcal{A}$ corresponds a mapping class from $M_{g,0}$, and this correspondence is a homomorphism, consequently $\mathcal{A}$ is represented on the space of genus g characters. But it follows from the results above that this representation is completely determined by the modular representation on the space of genus 1 characters, because we know the traces of all the representation operators. In other words, the space of genus g characters affords representations of the automorphism groups of all genus g closed surfaces, and these representations are completely determined by genus 1 data. This puts severe arithmetic restrictions on the allowed values of twisted dimensions, and hence on modular matrix elements, which might prove useful in classification attempts.

As an example, let us consider the Klein quartic [14], i.e. the surface of genus 3 with the maximum number (=168) of automorphisms allowed by Hurwitz's theorem. The

Table 2
Conjugacy classes of $\mathcal{A}$ and the corresponding traces

Order	Cardinality	No. of fixed points	Trace
2	21	4	$\mathcal{D}_1 \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$
3	56	2	$\mathcal{D}_1 \begin{pmatrix} 0 & 0 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$
4	42	0	$\mathcal{D}_1 \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$
7	48	3	$\mathcal{D}_1 \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{7} & \frac{2}{7} & \frac{4}{7} \end{pmatrix}$

automorphism group $\mathcal{A}$ is isomorphic to SL(3, 2), and its non-trivial elements are listed in Table 2.

According to the table, there is, for example, 21 involutions, i.e. elements of order 2 in $\mathcal{A}$, which form a single conjugacy class, each of them has four fixed points on the Klein quartic, and according to the theory developed above, the trace of the operator representing the corresponding mapping class on the space of genus 3 characters is given by the expression

$$\mathcal{D}_1 \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

The knowledge of the character table of $\mathcal{A}$ combined with the above table allows us to deduce, e.g. the conditions

$$\mathcal{D}_3 - \mathcal{D}_0 \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{7} & \frac{2}{7} & \frac{4}{7} \end{pmatrix} \in 7\mathbb{Z}_+, \qquad \mathcal{D}_3 - \mathcal{D}_1 \begin{pmatrix} 0 & 0 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \in 3\mathbb{Z}_+,$$

$$\mathcal{D}_3 - \mathcal{D}_1 \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \in 4\mathbb{Z}_+$$

with $\mathcal{D}_3$ denoting the number of genus 3 characters. These non-trivial congruences should hold in any consistent RCFT. For example, in the case of the Ising model, we find that the representation of $\mathcal{A}$ on the space of genus 3 characters contains four copies of the trivial representation, four copies of the six-dimensional irrep, and one copy of the eight-dimensional irrep.

In summary, we have found that traces of mapping classes of finite order are determined by the modular representation through Verlinde-like formulae, which in the $g = 1$ case lead to interesting trace identities for modular matrix elements. The origin of these trace formulae is the overlap between the classes of Seifert- and that of fibred 3-manifolds. The representation of the automorphism groups of surfaces on the space of characters is also

determined by the genus 1 data, and this leads to interesting arithmetic restrictions on the allowed modular representations in consistent RCFTs.

Acknowledgements

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